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LETTER TO THE EDITOR

Banach * algebraic bundles and algebraic generator coordinate method

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Abstract. A generalization of the AGCM method by using Banach * algebraic bundles over locally compact group was presented.

In this letter we propose another view on the algebraic generator coordinate method (AGCM) [1, 2]. We analyse obtained results, i.e. the algebraic generalization of the GCM method in the case of locally compact group, looking for the geometrical and algebraic structures standing behind this approach. The underlying formalism is the elegant, geometric language of a Banach * algebraic bundle $\mathscr{B}$. In the cross-sectional space of the bundle one can introduce in a natural way the notion of convolution and involution. Then applying the standard Gel'fand–Naimark–Segal construction (GNS) one obtains the space of states generated by the group G and the positive linear functional on L^1 (cross-sectional space of $\mathscr{B}$).

A Banach * algebraic bundle $\mathscr{B}=\langle B, \pi, \cdot, *\rangle$ over the locally compact and, for simplicity, unimodular group G will be the arena for our considerations. In the system $\mathscr{B}$, π is a continuous open surjection from the bundle space B into the base space G, B is a topological space furnished with appropriate operations and norms converting each fibre $\pi^{-1}(x)$ into a Banach space. The product $\cdot$ is bilinear and associative, satisfying the condition $\pi^{-1}(x)\cdot\pi^{-1}(y)\subset\pi^{-1}(x\cdot y)$ for all $x, y\in G$. The symbol * denotes an operation of involution on B. More information concerning this structure can be found in [3].

It is known, by the theorem of Douady and dal Soglio–Herault [3], that the bundle $\mathscr{B}$ has enough continuous cross sections and we can form L^1 cross-sectional space of $\mathscr{B}$ with respect to the Haar measure $d\mu$ on G. This space will consist of all those locally $d\mu$-measurable cross-sections of $\mathscr{B}$ which vanish outside a countable union of compact sets and for which the following condition is fulfilled:

$$\|f\|_1=\int_G \|f(x)\|\,d\mu<\infty. \tag{1}$$

Two elements of $L^1(d\mu, \mathscr{B})$ will be identified if they differ only on $d\mu$-nullset.

Let us suppose that f and g belong to a dense subspace $\mathscr{L}(\mathscr{B})$, of $L^1(d\mu, \mathscr{B})$. Then we fix $x\in G$. Continuity of multiplication in B together with the condition, which results from the definition of $\mathscr{B}$, i.e. $\pi(b\cdot c)=\pi(b)\pi(c)$ for all $b, c\in B$, shows that the map $y\to f(y)g(y^{-1}x)$ is continuous from G to $\pi^{-1}(x)$. It reveals the fact that equation

$$(f\circ g)(x)=\int_G f(y)g(y^{-1}x)\,d\mu \tag{2}$$

makes sense as $\pi^{-1}(x)$ valued integral. If x is allowed to vary the above equation defines $(f \circ g)(x)$ as a cross-section of $\mathscr{B}$. The convolution $f \circ g$ is linear with respect to f and g and it is associative. Since $\mathscr{L}(\mathscr{B})$ is dense in $L^1(\mathrm{d}\mu, \mathscr{B})$ the product $\circ$ in $\mathscr{L}(\mathscr{B})$ can be uniquely extended to a product $\circ$ in $L^1(\mathrm{d}\mu, \mathscr{B})$. In this way we have obtained $L^1(\mathrm{d}\mu, \mathscr{B})$ algebra of fibre valued functions on the group G with convolution as a multiplication law. It is called L^1 cross-sectional algebra of $\mathscr{B}$ [3]. One can define involution in $L^1(\mathrm{d}\mu, \mathscr{B})$ as follows:

$$f^{*}(x) = (f(x^{-1}))^{*} \tag{3}$$

where $f \in L^1(\mathrm{d}\mu, \mathscr{B})$, $x \in G$ and $*$ is the involution in the Banach * algebraic bundle $\mathscr{B}$.

It is shown [3] that for every f, g pertaining to the space $L^1(\mathrm{d}\mu, \mathscr{B})$ the involution (3) fulfils the following conditions:

(1) $\|f^{\#}\|_1 = \|f\|_1 < \infty$;
(2) $(f \circ g)^{\#}(x) = (g^{\#} \circ f^{\#})(x)$;
(3) $(f^{\#})^{\#} = f$;
(4) if $f \in L^1(\mathrm{d}\mu, \mathscr{B})$ then $f^{\#} \in L^1(\mathrm{d}\mu, \mathscr{B})$.

It implies that $L^1(\mathrm{d}\mu, \mathscr{B})$ cross-sectional algebra, equipped with the involution # and convolution $\circ$, is a Banach # algebra. For a group bundle (i.e. a trivial Banach algebraic bundle $B = C \times G$, where C is the algebra of complex numbers) the L^1 cross-sectional algebra is just L^1 group algebra over the group G used in the AGCM method.

In the AGCM method, after having introduced the involution and convolution in the Banach algebra of $L^1(G)$ functions, one has to introduce the complex positive functions on G. Then by means of them we define a positive linear functional on the $L^1(G)$ algebra [2]. In the case of Banach * algebraic bundle formulation of the AGCM method we introduce on $\mathscr{B}$ a linear functional, i.e. a function $\rho : B \to C$ which restriction to the fibre $\pi^{-1}(x)$ is linear for each $x \in G$. In addition ρ is continuous on B and its norm is continuous on each fibre $\pi^{-1}(x)$. We can form a norm of ρ restricted to each fibre $\pi^{-1}(x)$. This norm is given by [3]:

$$\|\rho\| = \sup\{\|\rho|_{\pi^{-1}(x)}\| : x \in G\} < \infty.$$

Having the notion of a linear functional on $\mathscr{B}$ we introduce the functional of positive type on $\mathscr{B}$, i.e. a continuous linear functional ρ satisfying the inequality

$$\sum_{i,j} \rho(b_j^{*} b_i) \geq 0$$

for finite sequence $b_1, \ldots, b_n$ of elements of B. In the case of a group bundle of G, where $b \in B$ is of the form $b = (c, x)$ with $x \in G$ and $c \in C$, the above equation reduces to the form

$$\sum_{i,j} \lambda_i \lambda_j^{*} \langle \rho ; x_j^{-1} x_i \rangle \geq 0$$

where $\langle \rho ; x \rangle = \rho(1, x)$, $\langle \rho, \cdot \rangle : G \to C$. It is called a function of positive type on G.

If $\mathscr{B}$ has an approximate unit $\{u_i\}$ then every functional ρ of positive type on $\mathscr{B}$ is [3]

(1) bounded
(2) $\rho(b^{*}) = (\rho(b))^{*}$.

In the particular case of a function of positive type on G it is bounded and satisfies $\langle \rho ; x^{-1} \rangle = \langle \rho ; x \rangle^{*}$ for every $x \in G$.

It turns out that the functional of positive type may be chosen in the form which arises from the so called *-representations [3].

The map $\rho: B \to C$ is given by

$$\rho(b) = (\xi, T_b \xi) \tag{4}$$

where ξ belongs to a Hilbert space $\mathscr{H}(T)$ which is the carrier space of *-representation of $\mathscr{B}$.

By a *-representation of $\mathscr{B}$, we mean a map T carrying B into the space of all continuous linear operators on $\mathscr{H}$ and satisfying the following conditions:

(1) T is linear on each fibre $\pi^{-1}(x)$;

(2) $T_b T_c = T_{bc}$ for all $b, c \in B$;

(3) $(T_b)^* = T_{b^*}$, $b \in B$ (* on the left side of this equation denotes the Hermitian conjugation in $\mathscr{H}(T)$);

(4) $b \to T_b \xi$ is continuous on B to $\mathscr{H}$, for all $\xi \in \mathscr{H}$.

When $\mathscr{B}$ has an approximate unit every functional p on $\mathscr{B}$ is of this form for some cyclic *-representation T of $\mathscr{B}$ and some cyclic vector ξ for T [3, 4].

Having the above functional on $\mathscr{B}$ we can extend the notion of linear functional to a linear functional on $L^1(\mathrm{d}\mu, \mathscr{B})$ as:

$$\langle \rho; f \rangle = \int_G \rho(f(x))\, \mathrm{d}\mu \tag{5}$$

where $f \in L^1(\mathrm{d}\mu, \mathscr{B})$. By straightforward calculations it can be easily proved that $\langle \rho; f \rangle$ is a positive linear functional on $L^1(\mathrm{d}\mu, \mathscr{B})$ and fulfils

$$|\langle \rho; f \rangle| = \left| \int_G \rho(f(x))\, \mathrm{d}\mu \right| \leq \int_G |\rho(f(x))|\, \mathrm{d}\mu \leq \|\rho\| \int_G \|f(x)\|\, \mathrm{d}\mu = \|\rho\|\, \|f\|_1.$$

This functional will be necessary in the construction of the space of states by means of the GNS procedure.

The first step in the GNS construction is to determine the left ideal I of 'zero' elements which causes the functional $\langle \rho; \rangle$ to have pathological behaviour:

$$I = \{g \in L^1(\mathrm{d}\mu, \mathscr{B}); \langle \rho; f^* \circ g \rangle = 0, \text{ for all } f \in L^1(\mathrm{d}\mu, \mathscr{B})\}. \tag{6}$$

Applying the Schwartz-Cauchy inequality:

$$|\langle \rho; f^* \circ g \rangle|^2 \leq \langle \rho; f^* \circ f \rangle \langle \rho; g^* \circ g \rangle \tag{7}$$

we can see that $f \in I$ iff $\langle \rho; f^* \circ f \rangle = 0$. Having found the left-ideal one may construct a quotient space $\mathscr{K} = L^1(\mathrm{d}\mu, \mathscr{B})/I$, which is a Hilbert space with the inner product given by:

$$(\mathrm{cl}_{\mathscr{K}}(f), \mathrm{cl}(g)_{\mathscr{K}})_{\mathscr{K}} = \langle \rho; f^* \circ g \rangle. \tag{8}$$

The legitimacy of the definition of $(,)_{\mathscr{K}}$ follows from the fact that $\langle \rho; f^* \circ g \rangle = \langle \rho; g^* \circ f \rangle^*$, here * is the usual complex conjugation.

More precisely, in the above construction we should use the dense subset $\mathscr{L}(\mathscr{B})$ of $L^1(\mathrm{d}\mu, \mathscr{B})$ instead of $L^1(\mathrm{d}\mu, \mathscr{B})$ and then complete the space, but this is the standard procedure which only complicates the language.

In this way we have shown that the construction presented in [2] can be generalized and described in terms of Banach * algebraic bundles and a positive linear functional defined on its cross-sectional space. This functional generates the Hilbert space of states. The flexibility of the construction allows for a natural description of the physical systems with varying physical state space. The formalism also generalizes the AGCM method to the case of vector-valued functions. For the standard generator coordinate method (GCM) this case corresponds to the vector-valued weight functions in the generator function which is a useful extension of the formalism in its physical applications.

Finally we remark that the C^*-algebraic quantization proposed by Landsman [5] can be roughly regarded as a special case of choosing the * algebraic Banach bundle, i.e. τ-semidirect product of Banach * algebra A and a group G [3]. If one chooses A as a C^*-algebra, then τ-semidirect product of A and G is a C^*-algebraic bundle. On the other hand this semidirect product bundle can be defined in terms of C^*-dynamical system, as a triple (A, G, τ). Having C^*-dynamical system the constructions presented in [5] can be conducted.

References

[1] Bogusz A and Góźdź A 1992 *J. Phys. A: Math. Gen.* **25** 4613

[2] Góźdź A and Rogatko M 1992 *J. Phys. A: Math. Gen.* **25** 4625

[3] Fell J M G and Doran R S *Representations of * Algebras, Locally Compact Groups and Banach *-Algebraic Bundles* 1988 (San Diego: Academic Press) vol I, II

[4] Pedersen G K 1979 *C*-Algebras and Their Automorphism Groups* (London: Academic Press)

[5] Landsman N P 1990 *Lett. Math. Phys.* **20** 11; 1990 *Rev. Math. Phys.* **2** 45, 73; 1991 *Nucl. Phys.* B **365** 121